Partial Solution Set, Leon Section 5.2

5.2.3 (a) Let $S = \text{Span}(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$. Let A be the matrix with rows $\mathbf{x}^T$ and $\mathbf{y}^T$. Show that $S^\perp = N(A)$.

Proof: By definition of S and A, we know that $S = RS(A)$. It follows that $S^\perp = (RS(A))^\perp = N(A)$.

(b) Find the orthogonal complement of the subspace of R^3 spanned by $(1, 2, 1)^T$ and $(1, -1, 2)^T$.

Solution: Based on part (a), we may let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$. The subspace spanned by the given vectors is simply $RS(A)$. So we want $N(A)$. Computing a basis for $N(A)$ in the usual way, we find that $N(A) = \text{Span}(-5, 1, 3)^T$. (When computing an arbitrary nullspace vector from the reduced matrix, you might have found something like $\mathbf{x} = (-5s/3, s/3, s)$, but don't forget that any multiple of $\mathbf{x}$ will serve.

5.2.4 Let S be the subspace of $\mathbf{R}^4$ spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T$ and $\mathbf{x}_2 = (0, 1, 3, 2)^T$. Find a basis for $S^\perp$.

Solution: The simplest solution is to construct a matrix A, with rows $\mathbf{x}_1^T$ and $\mathbf{x}_2^T$, and to then find a basis for $N(A) = RS(A)^\perp$. Details are suppressed, but one basis for $N(A)$ is given by $\mathbf{x}_3 = (2, -3, 1, 0)^T$ and $\mathbf{x}_4 = (1, 2, 0, -1)^T$.

5.2.5 Let $P_1 = (1, 1, 1), P_2 = (2, 4, -1)$, and $P_3 = (0, -1, 5)$.

(a) Find a nonzero vector $\mathbf{N}$ that is orthogonal to $\overline{P_1P_2}$ and to $\overline{P_1P_3}$.

Solution: Let $\overline{P_1P_2}$ and $\overline{P_1P_3}$ be the rows of a matrix A, and find a vector from $N(A)$. One such vector is $\mathbf{x} = (8, -2, 1)^T$.

(b) Find the equation of the plane determined by the three points.

Solution: We may choose any of the three given points to assume the role of P_0. Choosing P_1, we get $8(x - 1) - 2(y - 1) + (z - 1) = 0$. Choosing P_2, we get $8(x - 2) - 2(y - 4) + (z + 1) = 0$. Choosing P_3, we get $8x - 2(y + 1) + (z - 5) = 0$. Each of the three simplifies to $8x - 2y + z = 7$.

5.2.6 Is it possible for a matrix to have the vector $(3, 1, 2)$ in its rowspace and $(2, 1, 1)^T$ in its nullspace? Explain.

Solution: No, it is not possible. The given vectors are not orthogonal. If this is unclear, reread Theorem 5.2.1 and its proof.

5.2.7 Let $\mathbf{a}_j$ be a nonzero column of an $m \times n$ matrix A. Is it possible for $\mathbf{a}_j$ to be in $N(A^T)$?

Solution: Certainly not. If it were a zero column, yes, but we are told otherwise. The only vector that lies in $CS(A) \bigcap N(A^T)$ is the zero vector.

5.2.11 Prove: If A is an $m \times n$ matrix and $\mathbf{x} \in \mathbf{R}^n$, then either $A\mathbf{x} = \mathbf{0}$ or there exists $\mathbf{y} \in RS(A)$ such that $x^T y \neq 0$.

Proof: It might be useful to draw the same diagram used in class while discussing Corollary 5.2.5. Suppose that $\mathbf{x} \in \mathbf{R}^n - N(A)$. Since $(RS(A))^{\perp} = N(A)$ and $\mathbf{x} \notin N(A)$, it follows from the definition of orthogonal complements that there must be at least one vector $\mathbf{y} \in RS(A)$ for which $\mathbf{x}^T\mathbf{y} \neq 0$. $\square$

MA/Ra, November 3, 2004